Equation for Simulation of Single Bunch Instabilities

Betatron Motion with Transverse Wake Field

$$\eta_i = \frac{x_i}{\sqrt{\beta}}, \quad \theta = \frac{1}{\nu_0}\int^s \frac{ds'}{\beta}$$

$$\frac{d^2\eta_i}{d\theta^2} + (\nu_0 + \Delta\nu_i)^2 \eta_i = \nu_0{}^2 \beta^{\frac{3}{2}} \frac{F_i}{E}$$

$$F_i = e\sum_{j=1}^{N_p} q_j x_j \frac{d}{ds} W^{\perp}(z_j - z_i, s) = e\,\beta^{\frac{1}{2}} \sum_{j=1}^{N_p} q_j \eta_j \frac{d}{ds} W^{\perp}(z_j - z_i, s)$$

Approximation Slowly Varying Amplitude And Phase

$$\left|\frac{d^2 a_i}{d\theta^2}\right| << \left|2 i\nu_0 \frac{da_i}{d\theta}\right|$$

$$\int \beta^{1/2} f\, d\theta'$$

$$\eta_i = \mathrm{Re}\left[a_i(\theta)\, e^{i\nu_0\theta}\right] = \frac{1}{2}\left(a_i(\theta)\, e^{i\nu_0\theta} + c.c\right)$$

$$F_i = \mathrm{Re}\left[f_i(\theta)\, e^{i\nu_0\theta}\right] = \frac{1}{2}\left(f_i(\theta)\, e^{i\nu_0\theta} + c.c\right)$$

$$f_i(\theta) = e\,\beta^{\frac{1}{2}} \sum_{j=1}^{N_p} q_j a_j \frac{d}{ds} W^{\perp}(z_j - z_i, s)$$

Averaging for one betatron wavelength

$$\frac{da_i}{d\theta} = i\Delta\nu\, a_i + \frac{\nu_0}{2iE_0}\int_{\theta-\frac{\pi}{2\nu_0}}^{\theta+\frac{\pi}{2\nu_0}} \beta^{\frac{3}{2}} f_i \frac{d\theta'}{\left(\frac{2\pi}{\nu_0}\right)} + \frac{\nu_0}{2iE_0}\int_{\theta-\frac{\pi}{2\nu_0}}^{\theta+\frac{\pi}{2\nu_0}} \beta^{\frac{3}{2}} f_i e^{-2i\nu_0\theta'} \frac{d\theta'}{\left(\frac{2\pi}{\nu_0}\right)}$$

Neglecting 2nd turm of R.H.S.

$$\int \beta^{3/2} f e^{-2i\nu_0} d\theta$$

$$\frac{da_i}{d\theta} = i\Delta\nu\, a_i + \frac{\nu_0}{2iE_0}\int_{\theta-\frac{\pi}{2\nu_0}}^{\theta+\frac{\pi}{2\nu_0}} \beta^{\frac{3}{2}} f_i \frac{d\theta'}{\left(\frac{2\pi}{\nu_0}\right)}$$

Using

$$\int_{\theta-\frac{\pi}{2\nu_0}}^{\theta+\frac{\pi}{2\nu_0}} \beta^{\frac{3}{2}} f_i \frac{d\theta'}{\left(\frac{2\pi}{\nu_0}\right)} = \int_{s-\frac{\lambda}{2}}^{s+\frac{\lambda}{2}} \beta^{\frac{1}{2}} f_i \frac{ds'}{2\pi} = \frac{1}{2\pi}\int_{s-\frac{C}{2}}^{s+\frac{C}{2}} \beta^{\frac{1}{2}} f_i\, ds' \frac{\lambda_\beta}{C}$$

$$\int_{s-\frac{C}{2}}^{s+\frac{C}{2}} \beta^{\frac{1}{2}} f_i\, ds' = \sum_k \beta_k{}^{\frac{1}{2}} \int_{k\text{-th element}} f_i\, ds' = e\sum_k \beta_k \sum_j q_j a_j W_k^{\perp}(z_j - z_i) = e\sum_j q_j a_j \sum_k \beta_k W_k^{\perp}(z_j - z_i)$$

$$\frac{\lambda_\beta}{C} = \frac{1}{\nu_0}.$$

Finally for Betatron Motion,

$$\frac{da_i}{d\theta} = i\,\Delta\nu_i\, a_i + \frac{e}{4i\pi E_0}\sum_{j=1}^{N_p} q_j a_j \sum_{k=1}^{N_l} \beta_k W_k^{\perp}(z_j - z_i)$$

$\Delta\nu(a_i)$: Amp. dep. $\Delta\nu$

$$\frac{da_i}{d\theta} = i\,\Delta\nu_i\,a_i + \frac{e}{4i\pi E_0}\sum_{j=1}^{N_p} q_j\,a_j \sum_{k=1}^{N_I} \beta_k\,W_k^{\perp}(z_j - z_i)$$

$$r = |a|, \quad a = re^{i\phi}\ , \qquad x = \sqrt{\beta}\,\eta_i = \sqrt{\beta}\,\mathrm{Re}\!\left[a_i(\theta)\,e^{i\nu_0\theta}\right]$$

Difference Equations for Simulation with Step $\Delta\theta = 2\pi\,\dfrac{\Delta T}{T_0}$

Suffix "- " means "before" and "+" means "after"

Lattice with Distributed Broad-Band Impedance

Transverse

Longitudinal

$$\Delta E_i^+ = \Delta E_i^- - U_0\,\frac{\Delta T}{T_0}\left(1 + \frac{\Delta E_i^-}{E_0}\right)^2$$

$$z_i^+ = z_i^- - \alpha\,\frac{\Delta E_i^-}{E_0}\,c\,\Delta T$$

Transverse

$$r_i^+ = r_i^- + \mathrm{Re}\!\left[\,f_i^-\,e^{-i\phi_{i,n}}\right]\Delta\theta$$

$$\phi_i^+ = \phi_i^- + \Delta\nu_i\,\Delta\theta + \frac{\mathrm{Im}\!\left[\,g_i^-\,e^{-i\phi_i^-}\right]}{\left[(r_i^+ + r_i^-)/2\right]}\,\Delta\theta$$

$$g_i^- = \frac{1}{4i\pi E_0}\sum_{j=1}^{N_p} q_j\,a_j^- \sum_{k=1}^{N_I} \beta_k\,W_k^{\perp}(z_j^- - z_i^-)$$

Localized Broad-Band Impedance

Longitudinal

$$\Delta E_i^+ = \Delta E_i^- - e\sum_{j=1}^{N_p} q_j\,W^{\parallel}(z_j^- - z_i^-)$$

Transverse

$$a_i^+ = a_i^- - \frac{e}{iE_0}\,\beta^{\frac{1}{2}}\sum_{j=1}^{N_p} q_j\,x_j^-\,W^{\perp}(z_j^- - z_i^-)$$

Acceleration

Longitudinal

$$\Delta E_i^+ = \Delta E_i^- + eV_c\sin\left(2\pi f_{rf}\,\frac{z_i^-}{c} + \phi_c\right)$$

$$eV_a(z) = eV_c\sin\left(2\pi f_{rf}\,\frac{z}{c} + \phi_c\right)$$

Transverse

$$a_i^+ = a_i^- - i\,\frac{eV_a^-}{E_0}\,\mathrm{Im}\!\left[a_i^- e^{i\nu_0\theta}\right]e^{-i\nu_0\theta}$$

Radiation Excitation

Longitudinal

$$\Delta E_i^+ = \Delta E_i^- + \sqrt{\left(4\,\frac{\Delta T}{\tau_E}\right)}\left(\frac{\sigma_{E,0}}{E_0}\right)u_i$$

Transverse

$$a_i^+ = a_i^- + \sqrt{\left(4\,\frac{\Delta T}{\tau_\beta}\,\varepsilon_0\right)}\;v_i\,e^{i\,2\pi\,w_i}$$